THE KING'S SCHOOL

2004
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown for every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value
- Start a new booklet for each question
- Put your Student Number and the question number on the front of each booklet

Total marks – 84
Attempt Questions 1-7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\dfrac{d}{dx}\left(e^{\tan x}\right)$. **2**

(b) The interval joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is trisected by the points $(-2,3)$ and $Q(1,0)$. Write down the coordinates of A and B. **3**

(c) Find the acute angle, to the nearest degree, between the lines $x - y = 2$ and $2x + y = 1$. **2**

(d) Use the substitution $u = 1 - x$ to evaluate $3\int_{-1}^{0} \dfrac{x}{\sqrt{1-x}}\, dx$. **3**

(e) For a given series $T_{n+1} - T_n = 7$ and $T_1 = 3$. Find the value of S_{100}, where $Sn = T_1 + T_2 + \ldots + T_n$. **2**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $\dfrac{x^2-2}{x}<1$. **3**

(b) Find

(i) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx$. **1**

(ii) $\displaystyle\int \frac{3}{5+x^2}\,dx$. **2**

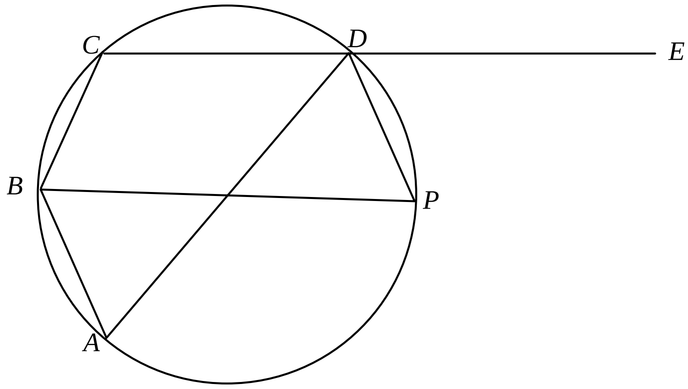
(c) Solve the equation $2\ln(3x+1)-\ln(x+1)=\ln(7x+4)$. **3**

(d) Solve $2\tan^3\theta-3\tan^2\theta-2\tan\theta+3=0$ for $0\le\theta\le 360°$, giving your answers to the nearest minute, where necessary. **3**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use one application of Newton's Method to approximate the root of the equation $e^x + x = 2$ which is near 0.5, correct to two decimal places. **3**

(b)



In the diagram above $ABCD$ is a cyclic quadrilateral. CD is produced to E. P is a point on the circle through A, B, C, D such that $\angle ABP = \angle PBC$.

(i) Copy the diagram showing the above information.

(ii) Explain why $\angle ABP = \angle ADP$. **1**

(iii) Show that PD bisects $\angle ADE$. **2**

(iv) If $\angle BAP = 90°$ and $\angle APD = 90°$, explain where the centre of the circle is located. **2**

(c) (i) Write $\cos x - \sqrt{3}\sin x$ in the form $A\cos(x + \alpha)$ where $A > 0$, $0 < \alpha < \pi$. **2**

(ii) Hence or otherwise, solve $\cos x - \sqrt{3}\sin x = 1$ for all values of x. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Find the polynomial $P(x)$, if $P(x)$ has

(α) degree 4;

(β) factors of $(x+3)^2$ and $(x-3)^2$; and

(γ) a remainder of –50 when divided by $x+2$. **2**

(ii) Sketch the curve. **1**

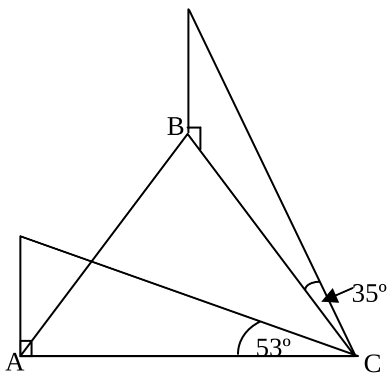
(b) The speed v cm/sec of a particle moving with simple harmonic motion in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where x cm is the magnitude of the displacement from a fixed point O.

(i) Show $\frac{d^2x}{dt^2} = -2(x-1)$. **2**

(ii) Find the period of the motion. **1**

(iii) Find the amplitude of the motion. **2**

(c) A and B are the feet of two towers of equal height. B lies due North of A. From a point C, 40m East of A and in the same horizontal plane, the angle of elevation of the top of the tower A is 53º. From the same point the angle of elevation of tower B is 35º. Find the distance between the towers, AB, correct to the nearest metre. **4**



Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

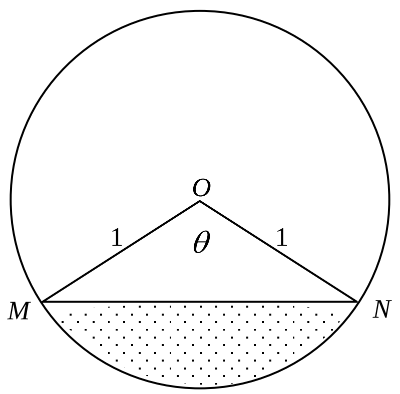
(a) For the function $y = 3\cos^{-1}\dfrac{x}{2}$

(i) Find the domain and the range. **2**

(ii) Sketch the curve. **1**

(iii) Find the equation of the tangent to the curve at the point on the curve where $x = 0$. **3**

(b) O is the centre of a circle, radius 1m and $\angle MON = \theta$ radians. The shaded segment formed by MN has an area A square metres and perimeter P metres.



(i) Prove $A = \dfrac{1}{2}(\theta - \sin\theta)$. **1**

and $P = \theta + 2\sin\dfrac{\theta}{2}$. **1**

(ii) P is increasing at a constant rate of R m/s. Find, in terms of R, the rate of increase of

(α) θ when $\angle MON = \dfrac{2\pi}{3}$; and **2**

(β) A when $\angle MON = \dfrac{2\pi}{3}$. **2**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the coordinates of the focus and the equation of the directrix of the parabola $x^2 = 4(x+y)$. **2**

(b) Prove by Induction that $3^{3n} + 2^{n+2}$ is divisible by 5 for all positive integers *n*. **4**

(c) Consider the variable point $T(-2t, t^2)$ on the parabola $y = \frac{1}{4}x^2$.

(i) Prove that the equation of the tangent at T is $y + tx + t^2 = 0$ **2**

(ii) If A is the x intercept of the tangent at T, find the coordinates of A. **1**

(iii) Find the coordinates of M, the midpoint of the interval TA. **1**

(iv) Find the equation in Cartesian form of the locus of the point M given in part (iii). **2**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A stone is thrown from the top of a building 15m high with an initial velocity of 26 m/s at an angle of $\tan^{-1}\frac{5}{12}$ to the horizontal.

If the acceleration due to gravity is 10m/sec^2, find

(i) the greatest height above the ground reached by the stone **2**

(ii) the time of flight **2**

(iii) the range of the stone **1**

(iv) the velocity after 2 seconds **2**

(b) Two of the roots of the equation $x^3 + ax^2 + b = 0$ are reciprocals of each other where a and b are real numbers.

Show that

(i) the third root is equal to $-b$; **1**

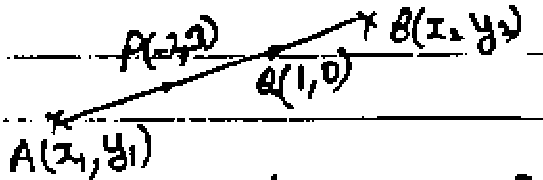
(ii) $a = b - \frac{1}{b}$; and **2**

(iii) the two roots, which are reciprocals, will be real if $-\frac{1}{2} \le b \le \frac{1}{2}$. **2**

End of Examination

1(a) $\sec^2 x \cdot e^{\tan x}$

(b)



A: $-2 = \frac{x_1+1}{2}$; $3 = \frac{y_1+0}{2}$

$x_1 = -5$; $y_1 = 6$

$\underline{A(-5,6)}$

B: $1 = \frac{-2+x_2}{2}$ $0 = \frac{3+y_2}{2}$

$x_2 = 4$ $y_2 = -3$

$\underline{B(4,-3)}$

(c) $l_1 : m_1 = 1$

$l_2 : m_2 = -2$

$\tan\theta = \left|\frac{1+2}{1+1\times -2}\right|$

$= 3$

$\theta = 72^\circ$

(d) $u = 1-x$ $x=-1, u=2$

$\frac{du}{dx} = -1$ $x=0, u=1$

$I = 3\int_2^1 \frac{1-u}{\sqrt{u}} \cdot -du$

$= 3\int_2^1 \frac{u-1}{u^{\frac{1}{2}}} du$

$= 3\int_2^1 u^{\frac{1}{2}} - u^{-\frac{1}{2}} . du$

$= 3\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_2^1$

$= 3\left[\frac{2}{3} - 2 - \left(\frac{2}{3}\times\sqrt{8} - 2\sqrt{2}\right)\right]$

$= 3\left[-\frac{4}{3} - \left(\frac{4}{3}\sqrt{2} - 2\sqrt{2}\right)\right]$

$= -4 - 4\sqrt{2} + 6\sqrt{2}$

$= \underline{2\sqrt{2} - 4}$

(e) $T_{n+1} - T_n = 7$

$T_1 = 3$

$T_2 - T_1 = 7$

$\therefore T_2 = 10$

$T_3 - T_2 = 7$

$T_3 = 17$

$T_4 - T_3 = 7$

$T_4 = 24$

$S_n = 3 + 10 + 17 + \cdots + T_n$

AS $a = 3$, $d = 7$

$S_{100} = \frac{100}{2}[2\times 3 + 99$

$= 50(6 + 693)$

$= 699 \times 50$

$= 34\,950$

Q2 (a) $\frac{x^2-2}{x} < 1$

c. Values: $x = 0$

$$x^2 - 2 = x$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2 \text{ or } -1$$

$-1 \quad 0 \quad 2$

test $x = -\frac{1}{2}$: $\frac{\frac{1}{4}-2}{-\frac{1}{2}} = \frac{-1\frac{3}{4}}{-\frac{1}{2}} = 3\frac{1}{2}$ false

$x = 1$: $\frac{1-2}{1} < 1$ true

$\therefore \quad x < -1$ or $0 < x < 2$

(b) (i) $\int \frac{e^{2x}}{1+e^{2x}}\,dx$

$$= \frac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}\,dx$$
$$= \frac{1}{2}\ln(1+e^{2x}) + C$$

(ii) $\int \frac{3}{5+x^2}\,dx$

$$= 3\int \frac{1}{5+x^2}\,dx$$
$$= \frac{3}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} + C$$

(c) $2\ln(3x+1) - \ln(x+1) = \ln(7x+4)$

$$\therefore \frac{(3x+1)^2}{x+1} = 7x+4$$

$$9x^2 + 6x + 1 = 7x^2 + 11x + 4$$
$$2x^2 - 5x - 3 = 0$$
$$(2x+1)(x-3) = 0$$
$$\therefore x = -\frac{1}{2} \text{ or } 3$$

but $3x+1 > 0$ & $x+1 > 0$ & $7x+4 > 0$

$x > -\frac{1}{3}$ & $x > -1$ & $x > -\frac{4}{7}$

$\therefore \quad x > -\frac{1}{3}$

$\therefore$ Solution $x = 3$

(d) $2\tan^3\theta - 3\tan^2\theta - 2\tan\theta + 3 =$

$\tan^2\theta(2\tan\theta - 3) - 1(2\tan\theta - 3)$

$\therefore (2\tan\theta - 3)(\tan^2\theta - 1) = 0$

$\therefore \tan\theta = \frac{3}{2}$ or ± 1

$\theta = 45°, 135°, 225°, 315°,$

$56°19', 236°19'.$

Q3 (a) $y = e^{x} + x - 2$

$\frac{dy}{dx} = e^{x} + 1$

let $x_1 = 0.5$

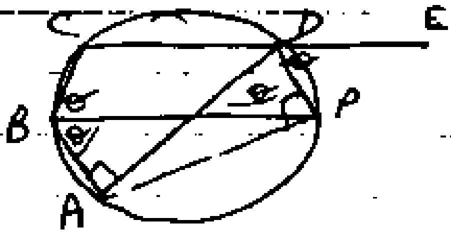
$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.5 - \frac{e^{0.5} + 0.5 - 2}{e^{0.5} + 1}$

$= 0.5 - \frac{0.148\ldots}{2.648\ldots}$

$= \underline{0.44}$

(b)



let $\angle ABP = \angle PBC = \theta$

(ii) $\angle ABP = \angle ADP = \theta$ (angles in same segment)

(iii) $\angle EDP = \angle PBC = \theta$ (ext. angle of cyclic quad = int. opp)

but $\angle ADP = \theta$ (from (ii))

$\therefore$ PD bisects $\angle ADE$

(iv) BP and AD are both diameters (angles in semi circle = 90°)

$\therefore$ centre of circle is the point of intersection of BP & AD.

(c) (i) $\cos x - \sqrt{3}\sin x \equiv A\cos(x+\alpha)$

$\equiv A[\cos x . \cos\alpha - \sin x$

$A\cos\alpha = 1$, $A\sin\alpha = \sqrt{3}$

$A = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\sin\alpha = \frac{1}{2}$

$\alpha = \frac{\pi}{3}$

$\therefore \cos x - \sqrt{3}\sin x = 2\cos(x +$

$\therefore 2\cos(x + \frac{\pi}{3}) = 1$

$\cos(x + \frac{\pi}{3}) = \frac{1}{2}$

$x + \frac{\pi}{3} = 2n\pi \pm \cos^{-1}(\frac{1}{2}$

$= 2n\pi \pm \frac{\pi}{3}$

$\therefore x = 2n\pi$ or $2n\pi -$

Q4 (a)(i) $P(x) = a(x+3)^2(x-3)^2$

$P(-2) = -50$

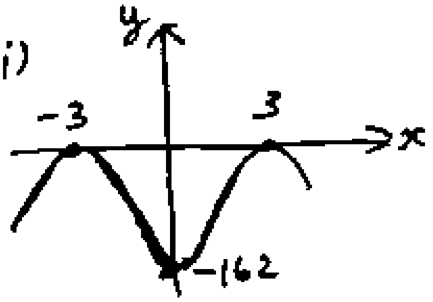
$\therefore -50 = a(1)^2(-5)^2$

$-50 = 25a$

$a = -2$

$P(x) = -2(x+3)^2(x-3)^2$

(ii)



(b)(i) $v^2 = 6+4x-2x^2$

$\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}(3+2x-x^2)$

$= 2-2x$

$= -2(x-1)$

(ii) $\ddot{x} = -n^2 x$

$n = \sqrt{2}$

$\therefore$ Period $= \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$ secs.

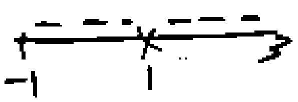
(iii) $v = 0$

$\therefore 2x^2-4x-6=0$

$x^2-2x-3=0$

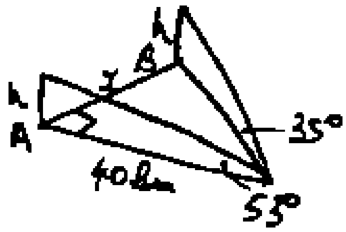
$(x-3)(x+1)=0$

$\therefore x = -1$ or 3



Amplitude = 2 cm.

(c)



$\tan 53° = \frac{h}{40}$

$h = 40\tan 53°$

$\tan 35° = \frac{h}{BC}$

$BC = \frac{h}{\tan 35°}$

$\therefore BC = \frac{40\tan 53°}{\tan 35°}$

$BC^2 = x^2 + 40^2$

$= \left(\frac{40\tan 53°}{\tan 35°}\right)^2 + 40$

$= 4146.94\ldots$

$x = 63.396\ldots$

$= 64$ m.

Q5 (a) $y = 3\cos^{-1}\frac{x}{2}$

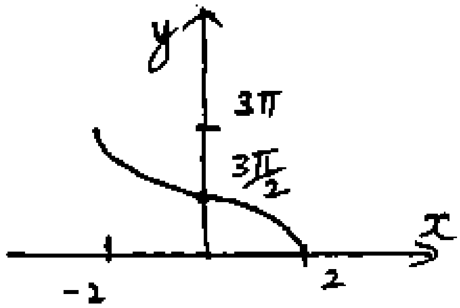
(i) $D: -1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$

$R: 0\times 3 \le y \le 3\times\pi$

$0 \le y \le 3\pi$

(ii)



(iii) $y = 3\cos^{-1}\frac{x}{2}$

$y' = \frac{-3}{\sqrt{1-\frac{x^2}{4}}} \times \frac{1}{2}$

at $x = 0$, $y' = \frac{-3}{2\sqrt{1}} = -\frac{3}{2}$

$y = \frac{3\pi}{2}$

eqn of tangent is

$y - \frac{3\pi}{2} = -\frac{3}{2}(x - 0)$

$2y - 3\pi = -3x$

$3x + 2y - 3\pi = 0$

(b) (i) $A = \frac{1}{2}\times 1\times 1\times\theta - \frac{1}{2}\times 1\times 1\times\sin\theta$

$= \frac{1}{2}(\theta - \sin\theta)$

$P = 1\times\theta + 2\times\sin\frac{\theta}{2}$

(ii) $R = \frac{dP}{dt}$

(α) $\frac{d\theta}{dt} = ?$

$P = \theta + 2\sin\frac{\theta}{2}$

$\frac{dP}{dt} = 1 + \cos\frac{\theta}{2}$

when $\theta = \frac{2\pi}{3}$, $\frac{dP}{dt} = 1 + \cos$

$=$

$\frac{d\theta}{dt} = \frac{d\theta}{dP} \times \frac{dP}{dt}$

$= \frac{2}{3} \times R$

$= \frac{2R}{3}$ radians/

(β) $\frac{dA}{dt} = ?$

$A = \frac{1}{2}(\theta - \sin\theta)$

$\frac{dA}{d\theta} = \frac{1}{2}(1 - \cos\theta)$

when $\theta = \frac{2\pi}{3}$ $\frac{dA}{d\theta} = \frac{1}{2}(1 - $

$= \frac{1}{2} \times$

$=$

$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= \frac{3}{4} \times \frac{2R}{3}$

$= \frac{R}{2}$ m^2/se

Q6 (a) $x^2 = 4x + 4y$

$x^2 - 4x + 4 = 4y + 4$

$(x-2)^2 = 4(y+1)$

Focus $(2,0)$

Directrix $y = -2$

(b) When $n=1$, $3^{3n} + 2^{n+2} = 3^3 + 2^3 = 35$

which is divisible by 5

assume true for $n = k$

i.e. $\dfrac{3^{3k} + 2^{k+2}}{5} = c$ (an integer)

$\therefore 3^{3k} = 5c - 2^{k+2}$

Prove true for $n = k+1$

i.e. $\dfrac{3^{3k+3} + 2^{k+3}}{5} = c_1$ (an integer)

$LHS = \dfrac{3^{3k}.3^3 + 2^{k+3}}{5}$

$= \dfrac{(5c - 2^{k+2}).27 + 2^{k+2}.2^1}{5}$

$= \dfrac{5 \times 27c - 27 \times 2^{k+2} + 2 \times 2^{k+2}}{5}$

$= \dfrac{5[27c - 5.2^{k+2}]}{5}$

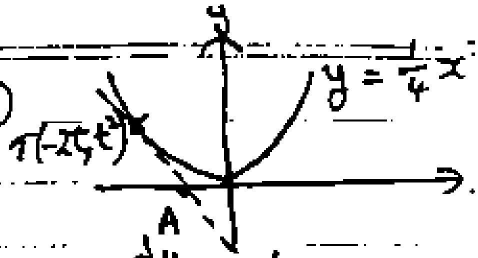
which is an integer if c is an integer

$\therefore$ if it is true for $n = k$, it is true for $n = k+1$

since it is true for $n=1$, it is true $n=2$

and so on for all n.

(c)



(i) $\dfrac{dy}{dx} = \frac{1}{2}x$

at T $\dfrac{dy}{dx} = -t$

eq'n of tangent at T

$y - t^2 = -t(x + 2t$

$y - t^2 = -tx - 2t$

$y + tx + t^2 = 0$

(ii) $A\left(-t, 0\right)$

(iii) $M\left(-\frac{3t}{2}, \frac{t^2}{2}\right)$

(iv) $x = -\frac{3t}{2}$, $y =$

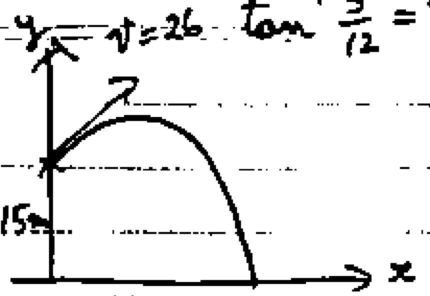
$t = -\frac{2x}{3}$

$\therefore y = \dfrac{\left(-\frac{2x}{3}\right)^2}{2}$

$= \frac{4x^2}{18}$

$y = \frac{2}{9}x^2$

Q 7 (a) $v = 26$ $\tan^{-1}\frac{5}{12} = \theta$



$t=0,\ x=0,\ y=15\ ,\ \dot{x} = 26\times\frac{12}{13} = 24\ ;\ \dot{y} = 26\times\frac{5}{13} = 10$

$\ddot{y} = -10$ $\qquad\qquad$ $\ddot{x} = 0$

$\dot{y} = -10t + c_1$ $\qquad\qquad$ $\dot{x} = c_3 = 24$

$t=0,\ \dot{y} = 10,\ c_1 = 10$

$\dot{y} = -10t + 10$

$y = -5t^2 + 10t + c_2$ $\qquad\qquad$ $x = 24t + c_4$

$t=0,\ y = 15,\ c_2 = 15$ $\qquad\qquad$ $t=0,\ x=0\ \ c_4 = 0$

$y = -5t^2 + 10t + 15$ $\qquad\qquad$ $x = 24t$

(i) greatest height $\dot{y} = 0,\ t = 1$

$y = -5 + 10 + 15$

$= 20m$

(ii) time of flight $y = 0$

$t^2 - 2t - 3 = 0$

$(t+1)(t-3) = 0$

$\therefore\ t = -1$ or 3

Time of flight = 3 secs

(iii) when $t = 3$ $\quad x = 24\times 3$

$= 72m$

(iv) $t = 2$ $\quad \dot{x} = 24,\ \dot{y} = 10 - 20 = -10$

$v = \sqrt{24^2 + (-10)^2}$

$= 26m$ per sec.

Q7 (b) $x^3 + ax^2 + b = 0$

let roots be α, $\frac{1}{\alpha}$ & β

$\therefore \alpha + \frac{1}{\alpha} + \beta = -a$ — ①

$\alpha \times \frac{1}{\alpha} + \alpha\beta + \frac{1}{\alpha} \times \beta = 0$ — ②

$\alpha \times \frac{1}{\alpha} \times \beta = -b$ — ③

(i) $\therefore \beta = -b$ from ③

(ii) $\alpha + \frac{1}{\alpha} - b = -a$ from ①

$\alpha + \frac{1}{\alpha} = b - a$

$1 - \alpha b - \frac{b}{\alpha} = 0$ from ②

$1 - b(\alpha + \frac{1}{\alpha}) = 0$

$\therefore 1 - b(b-a) = 0$

$1 = b(b-a)$

$\frac{1}{b} = b - a$

$a = b - \frac{1}{b}$.

(iii) $\alpha + \frac{1}{\alpha} = b - a$

$= b - b + \frac{1}{b}$

$\alpha + \frac{1}{\alpha} = \frac{1}{b}$

$\alpha^2 - \frac{1}{b}\alpha + 1 = 0$

real roots $\Delta \geq 0$

$\therefore \frac{1}{b^2} - 4 \geq 0$

$1 - 4b^2 \geq 0$

$b^2 \leq \frac{1}{4}$

$-\frac{1}{2} \leq b \leq \frac{1}{2}$